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Entanglement for a two-parameter class of states in $2 \otimes n$ quantum system

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Abstract

We exhibit a two-parameter class of states $\rho_{(\alpha,\gamma)}$ in a $2 \otimes n$ quantum system for $n \geqslant 3$, which can be obtained from an arbitrary state by means of local quantum operations and classical communication, and which are invariant under all bilateral operations on a $2 \otimes n$ quantum system. We calculate the negativity of $\rho_{(\alpha,\gamma)}$, and a lower bound and a tight upper bound on its entanglement of formation. It follows from this calculation that the entanglement of formation of $\rho_{(\alpha,\gamma)}$ cannot exceed its negativity.

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1. Introduction

Entanglement is one of the most important resources for quantum communication and information processing including quantum cryptography, teleportation and superdense coding. On this account, the research on entanglement has been considerably developed and has improved quantum information science in recent years. In particular, the quest for proper measures of entanglement has received a great deal of attention, and several measures of entanglement, such as the negativity and the entanglement of formation, have been proposed [1–5].

Peres–Horodeckis' criterion for separability [6, 7] leads to a natural entanglement measure, called the *negativity* $\mathcal{N}$, defined by

$$\mathcal{N}(\rho) = \|\rho^{T_B}\|_1 - 1 \tag{1}$$

where ρ^{T_B} is the partial transpose of a state ρ in a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\|\cdot\|_1$ is the trace norm. Vidal and Werner [8] defined the negativity of a state ρ as $(\|\rho^{T_B}\|_1 - 1)/2$, which corresponds to the absolute value of the sum of negative eigenvalues of ρ^{T_B}, and which vanishes for separable states. For a maximally entangled pure state such as one of the Bell states, this quantity is strictly less than one. In order for any maximally entangled pure state in a $2 \otimes n$ quantum system to have negativity one, it must be defined as equation (1).

We note that the negativity is an entanglement monotone [9] under local quantum operations and classical communication (LOCC) [8], and that it is a measure of entanglement which can be computed effectively for any state. However, although the positivity of the partial transpose is a necessary and sufficient condition for nondistillability in a $2 \otimes n$ quantum system [10, 11], there exist entangled states with positive partial transposition (PPT) in any bipartite system except for in $2 \otimes 2$ and $2 \otimes 3$ quantum systems [10, 12], and hence it is not sufficient for the negativity to be a good measure of entanglement even in a $2 \otimes n$ quantum system.

In this paper we consider $2 \otimes n$ quantum systems for $n \geqslant 3$, and exhibit a two-parameter class of states in a $2 \otimes n$ quantum system, which can be obtained from an arbitrary state by means of LOCC and are invariant under all unitary operations of the form $U \otimes U$ on a $2 \otimes n$ quantum system[3]. We show that a state in the two-parameter class has a PPT if and only if the state is separable, so that the negativity can be a measure to quantify the amount of entanglement of states in the class. We remark that a finite dimensional truncation of a single two-level atom interacting with a single-mode quantized field [13] can be regarded as a $2 \otimes n$ quantum system.

The *entanglement of formation* is defined to be the convex-roof extension of the pure-state entanglement, that is, the minimum average of the pure-state entanglement over all ensemble decompositions of a given state ρ,

$$E_f(\rho) = \min_{\sum_k p_k |\psi_k\rangle\langle\psi_k| = \rho} \sum_k p_k E(|\psi_k\rangle). \tag{3}$$

Here, the pure-state entanglement E is defined as the entropy of subsystem A, $E(|\psi\rangle) = S(\mathrm{tr}_B(|\psi\rangle\langle\psi|))$, S is the von Neumann entropy.

Since the concept of pure-state entanglement E has most widely been accepted as an entanglement measure for pure states and the entanglement of formation is its natural extension as seen in equation (3), the entanglement of formation is one of the best measures for bipartite quantum systems which are known so far. Nevertheless, there is no known explicit formula for the entanglement of formation of states in a general quantum system except for states in the $2 \otimes 2$ quantum system [2, 14], the isotropic states and the Werner states in a $n \otimes n$ quantum system [4, 15, 16] and states of the specific form [4, 5]. For a $2 \otimes n$ quantum system, only a lower bound on the entanglement of formation is given by decomposing a $2 \otimes n$ dimensional Hilbert space into many $2 \otimes 2$ dimensional subspaces [17, 18].

In this paper we present a lower bound and a tight upper bound on the entanglement of formation for the two-parameter class of states. The explicit calculations of the negativity and bounds on the entanglement of formation show that the entanglement of formation of any state in the two-parameter class cannot exceed its negativity.

This paper is organized as follows: in section 2 we exhibit a two-parameter class of states and a procedure involving only LOCC which transforms an arbitrary state into one of the states

[3] Let $U(k)$ be the group of all unitary operators on a k-dimensional Hilbert space, and $\{|0\rangle_A, |1\rangle_A\}$ and $\{|0\rangle_B, |1\rangle_B, \ldots, |n-1\rangle_B\}$ be bases of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. For convenience, we now identify a unitary operator $U_A \in \mathrm{U}(2)$ with $U_B \in \mathrm{U}(n)$ if for $j = 0, 1$, $U_A|j\rangle_A = a_j|0\rangle_A + b_j|1\rangle_A$ and $U_B|j\rangle_B = a_j|0\rangle_B + b_j|1\rangle_B$. For $0 < m < n$, we let

$$G(m,n) = \{U \in \mathrm{U}(n) : U(\mathcal{H}_m) = \mathcal{H}_m, U(\mathcal{H}_m^{\perp}) = \mathcal{H}_m^{\perp}\} \tag{2}$$

where $\mathcal{H}_m$ is a subspace of $\mathcal{H}_B$ generated by $|0\rangle_B, |1\rangle_B, \ldots, |m-1\rangle_B$, and $\mathcal{H}_m^{\perp}$ is the orthogonal complement of $\mathcal{H}_m$ in $\mathcal{H}_B$. Then $G(2,n)$ is a subgroup of $\mathrm{U}(n)$, and if U is a unitary operator in $G(2,n)$ then it is compatible to write a unitary operator of the form $U \otimes U$ on a $2 \otimes n$ quantum system.

in the class, and show that the states in the class are invariant under all unitary operations of the form $U \otimes U$. In section 3 we explicitly calculate the negativity, and a lower bound and a tight upper bound on the entanglement of formation for the two-parameter class, and compare its negativity with the entanglement of formation. Finally, in section 4 we summarize our results and discuss a generalization of the two-parameter class into a higher dimensional system.

2. A two-parameter class of states in a $2 \otimes n$ quantum system

We consider the following class of states with two real parameters α and γ in a $2 \otimes n$ quantum system:

$$\rho_{(\alpha,\gamma)} = \alpha \sum_{i=0}^{1} \sum_{j=2}^{n-1} |ij\rangle\langle ij| + \beta(|\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+|) + \gamma|\psi^-\rangle\langle\psi^-| \tag{4}$$

where $\{|ij\rangle : i = 0, 1, j = 0, 1, \ldots, n-1\}$ is an orthonormal basis for $2 \otimes n$ quantum system,

$$|\phi^\pm\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \tag{5}$$

$$|\psi^\pm\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \tag{6}$$

and the parameter β is dependent on α and γ by the unit trace condition,

$$2(n-2)\alpha + 3\beta + \gamma = 1. \tag{7}$$

From the unit trace condition in equation (7) we can readily obtain the domain for the parameters α and γ, $0 \leqslant \alpha \leqslant 1/2(n-2)$ and $0 \leqslant \gamma \leqslant 1$. We note that the states of the form $\rho_{(0,\gamma)}$ for $0 \leqslant \gamma \leqslant 1$ are equal to Werner states [19] in a $2 \otimes 2$ quantum system, that the states are entangled and distillable if and only if $1/2 < \gamma \leqslant 1$ and that for $1/2 \leqslant \gamma \leqslant 1$,

$$\mathcal{N}(\rho_{(0,\gamma)}) = 2\gamma - 1$$
$$E_f(\rho_{(0,\gamma)}) = h\left(\frac{1 + \sqrt{1 - \mathcal{N}(\rho_{(0,\gamma)})^2}}{2}\right) = h\left(\frac{1}{2} + \sqrt{\gamma \cdot (1-\gamma)}\right)$$

where h is the binary Shannon entropy.

We are going to show now that an arbitrary state ρ in a $2 \otimes n$ quantum system can be transformed to a state of the form $\rho_{(\alpha,\gamma)}$ in equation (4) by LOCC. In other words, we will show that there exist unitary operators U_k and probabilities p_k such that

$$\sum_k p_k (U_k \otimes U_k)\rho\left(U_k^\dagger \otimes U_k^\dagger\right) = \rho_{(\alpha,\gamma)} \tag{8}$$

for some $0 \leqslant \alpha \leqslant 1/2(n-2)$ and $0 \leqslant \gamma \leqslant 1$, using a method similar to those presented by Bennett *et al* [1] and Dür *et al* [11].

We define the operation U_θ as

$$U_\theta : |j\rangle \mapsto (e^{i\theta})^j |j\rangle$$

where $i = \sqrt{-1}$. We first perform $U_\pi \otimes U_\pi$ with probability 1/2, while with probability 1/2 no operation is performed, that is,

$$\tfrac{1}{2}(U_\pi \otimes U_\pi)\rho\left(U_\pi^\dagger \otimes U_\pi^\dagger\right) + \tfrac{1}{2}\rho. \tag{9}$$

Let us now define the operation U_k by $U_k : |j\rangle \mapsto (-1)^{\delta_{j,k}}|j\rangle$ for $k = 2, 3, \ldots, n-1$, and then for each $k = 2, 3, \ldots, n-1$, perform $U_k \otimes U_k$ with probability 1/2, while applying the identity operation with probability 1/2, respectively. Here, we remark that $U_k \otimes U_k = I \otimes U_k$. We now perform $U_{\pi/2} \otimes U_{\pi/2}$ with probability 1/2 as in equation (9), and then perform the swap operator $U_{01} : |0\rangle \leftrightarrow |1\rangle$ ($|j\rangle \mapsto |j\rangle$ for $2 \leqslant j \leqslant n-1$) with probability $1/2$. Then we obtain a state of the following form:

$$\sum_{j=2}^{n-1} a_j(|0j\rangle\langle 0j| + |1j\rangle\langle 1j|) + b(|\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-|) + c_+|\psi^+\rangle\langle\psi^+| + c_-|\psi^-\rangle\langle\psi^-|. \qquad (10)$$

Let T be the unitary operator defined by $|0\rangle \mapsto |0\rangle$, $|1\rangle \mapsto |1\rangle$, $|2\rangle \mapsto |3\rangle$, $|3\rangle \mapsto |4\rangle, \ldots,$ $|n-2\rangle \mapsto |n-1\rangle$ and $|n-1\rangle \mapsto |2\rangle$. Now we perform the following operation:

$$\rho \mapsto \frac{1}{n-2}\sum_{j=0}^{n-3}(T^j \otimes T^j)\rho(T^j \otimes T^j)^\dagger.$$

Here, we also remark that $T^j \otimes T^j = I \otimes T^j$ for any $j = 0, 1, \ldots, n-3$. Then a state in equation (10) now has the form

$$a\sum_{i=0}^{1}\sum_{j=2}^{n-1}|ij\rangle\langle ij| + b(|\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-|) + c_+|\psi^+\rangle\langle\psi^+| + c_-|\psi^-\rangle\langle\psi^-|.$$

Let H be the unitary operator defined as $|0\rangle \mapsto (|0\rangle + |1\rangle)/\sqrt{2}$, $|1\rangle \mapsto (|0\rangle - |1\rangle)/\sqrt{2}$ and $|j\rangle \mapsto |j\rangle$ for $2 \leqslant j \leqslant n-1$. After performing as follows:

$$\rho \mapsto \tfrac{2}{3}(H \otimes H)\rho(H \otimes H) + \tfrac{1}{3}\rho$$

let us perform the sequence of the previous operations again. Then one can easily check that one of the states with two parameters in equation (4) is obtained, that is, there exist unitary operators U_k and probabilities p_k satisfying equation (8), and that furthermore if a state ρ is given by

$$\rho = \sum_{i=0}^{1}\sum_{j=2}^{n-1} a_{ij}|ij\rangle\langle ij| + b_+|\phi^+\rangle\langle\phi^+| + b_-|\phi^-\rangle\langle\phi^-| + c_+|\psi^+\rangle\langle\psi^+| + c_-|\psi^-\rangle\langle\psi^-| + \cdots$$

then the two-parameter state transformed by the above procedure becomes $\rho_{(\alpha,\gamma)}$ in equation (4) where $\alpha = \sum_{i,j} a_{ij}/(2n-4)$ and $\gamma = c_-$.

Noting that $(U \otimes I)|\psi^-\rangle = (I \otimes \pm U)|\psi^-\rangle$ for any U in the group $G(2,n)$ defined in equation (2), we can readily show that $\rho_{(\alpha,\gamma)}$ is invariant under all $U \otimes U$, that is, for any $U \in G(2,n)$,

$$(U \otimes U)\rho_{(\alpha,\gamma)}(U^\dagger \otimes U^\dagger) = \rho_{(\alpha,\gamma)}. \qquad (11)$$

We now define the $(U \otimes U)$-*twirling* superoperator $\mathcal{T}$ as

$$\mathcal{T}(\rho) = \int_{G(2,n)} \mathrm{d}\mu_U (U \otimes U)\rho(U^\dagger \otimes U^\dagger)$$

and $\mathrm{d}\mu_U$ is the normalized Haar measure on $G(2,n)$. Then it follows from equations (8) and (11) that $\mathcal{T}(\rho) = \rho_{(\alpha,\gamma)}$ for some α and γ, and that $\mathcal{T}(\rho_{(\alpha,\gamma)}) = \rho_{(\alpha,\gamma)}$, respectively. We remark that the negativity and the entanglement of formation are entanglement monotones [8, 9]. Thus, it follows that the negativity and the entanglement of formation for a given ρ are not less than those for $\mathcal{T}(\rho)$, respectively.

3. Entanglement for the states with two parameters

In this section, we consider two measures of entanglement for $\rho_{(\alpha,\gamma)}$, the negativity and the entanglement of formation, and explicitly calculate the value of its negativity and bounds on its entanglement of formation.

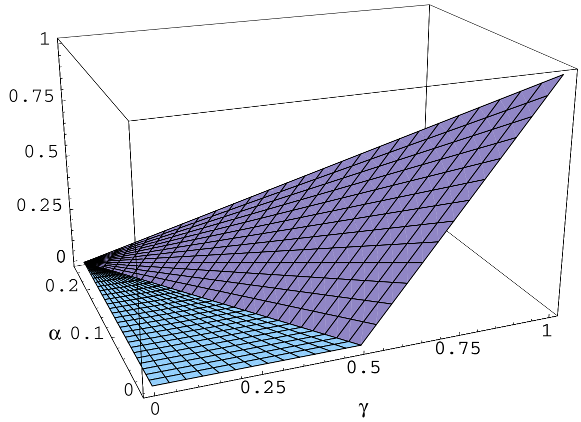

Figure 1. The negativity of $\rho_{(\alpha,\gamma)}$ in a $2 \otimes 4$ quantum system.

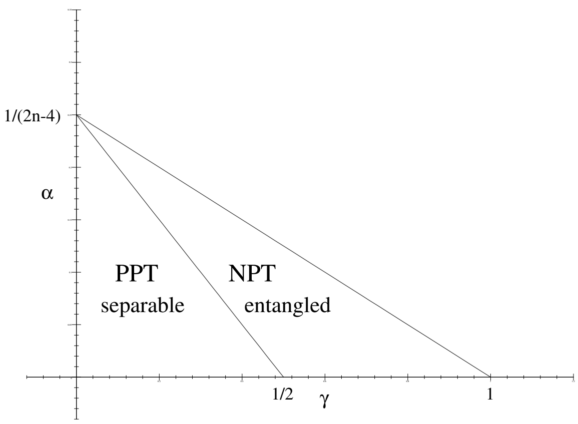

Figure 2. The domain of the parameters α and γ for the states $\rho_{(\alpha,\gamma)}$. All states in the PPT region are separable and undistillable, and all states in the NPT region are entangled and distillable.

In order to calculate the negativity of $\rho_{(\alpha,\gamma)}$, we have to compute its partial transpose:

$$\rho_{(\alpha,\gamma)}^{T_B} = \alpha \sum_{i=0}^{1} \sum_{j=2}^{n-1} |ij\rangle\langle ij| + \frac{\beta+\gamma}{2}(|01\rangle\langle 01| + |10\rangle\langle 10| + |\phi^-\rangle\langle\phi^-|) + \frac{3\beta-\gamma}{2}|\phi^+\rangle\langle\phi^+|.$$

Since $(3\beta - \gamma)/2 = 1/2 - (n-2)\alpha - \gamma \geqslant 0$ if and only if $\rho_{(\alpha,\gamma)}^{T_B}$ is positive, the negativity of $\rho_{(\alpha,\gamma)}$ is $\max\{(2n-4)\alpha + 2\gamma - 1, 0\}$, whose graph is shown in figure 1. Then the domain of the parameters α and γ for $\rho_{(\alpha,\gamma)}$ consists of two triangular regions, the PPT region satisfying $0 \leqslant (n-2)\alpha + \gamma \leqslant 1/2$ and the nonpositive partial transposition (NPT) region satisfying $1/2 < (n-2)\alpha + \gamma \leqslant 1$, as shown in figure 2. We note that all states in the PPT region are separable since three states, $\rho_{(0,0)}$, $\rho_{(0,1/2)}$ and $\rho_{(1/(2n-4),0)}$, corresponding to the vertices of the PPT region, are separable and all states in the region are convex combinations of the three states.

We now consider the entanglement of formation for $\rho_{(\alpha,\gamma)}$. Even though it is not known whether the explicit formula of the entanglement of formation for states in a $2 \otimes n$ quantum

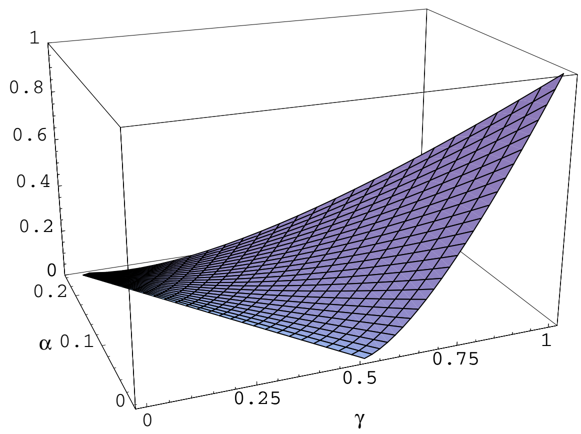

Figure 3. The lower bound (13) on the entanglement of formation of $\rho_{(\alpha,\gamma)}$ for $1/2 \leqslant (n-2)\alpha + \gamma \leqslant 1$ in a $2 \otimes 4$ quantum system.

system can be computed or not, one can readily compute one of its lower bounds [17, 18],

$$\mathcal{E}\left(\sqrt{\sum_{i<j} C_{ij}^2}\right) \tag{12}$$

where

$$\mathcal{E}(c) = h\left(\frac{1+\sqrt{1-c^2}}{2}\right)$$

C_{ij} is the Wootters concurrence [2] in a $2 \otimes 2$ dimensional subsystem which is supported by the bases $|0i\rangle$, $|1i\rangle$, $|0j\rangle$ and $|1j\rangle$. It is straightforward to check that the lower bound in equation (12) is

$$\begin{aligned}\mathcal{E}(\mathcal{N}(\rho_{(\alpha,\gamma)})) &= h\left(\frac{1+\sqrt{1-\mathcal{N}(\rho_{(\alpha,\gamma)})^2}}{2}\right)\\ &= h\left(\frac{1}{2}+\sqrt{((n-2)\alpha+\gamma)(1-(n-2)\alpha-\gamma)}\right)\end{aligned} \tag{13}$$

since $C_{01} = \mathcal{N}(\rho_{(\alpha,\gamma)})$ and $C_{ij} = 0$ unless $i = 0$ and $j = 1$. For a $2 \otimes 4$ quantum system, the lower bound on $E_f(\rho_{(\alpha,\gamma)})$ is plotted in figure 3.

In order to obtain a tight upper bound of $E_f(\rho_{(\alpha,\gamma)})$, we consider two-parameter states satisfying $2(n-2)\alpha + \gamma = 1$, that is, the states corresponding to the line through two points $(1, 0)$ and $(0, 1/(2n-4))$ in figure 2. Then the states are of the following form:

$$\begin{aligned}\rho_{(\alpha,\gamma)} &= \alpha \sum_{i=0}^{1}\sum_{j=2}^{n-1} |ij\rangle\langle ij| + \gamma|\psi^-\rangle\langle\psi^-|\\ &= \frac{1-\gamma}{2(n-2)} \sum_{i=0}^{1}\sum_{j=2}^{n-1} |ij\rangle\langle ij| + \gamma|\psi^-\rangle\langle\psi^-|\\ &\equiv \varrho_\gamma.\end{aligned} \tag{14}$$

We note that equation (14) is an eigenvalue decomposition of a state ϱ_γ, and that

$$E_f(\varrho_\gamma) \leqslant \gamma \tag{15}$$

in view of the convexity of E_f. Let $\varrho_\gamma = \sum_k p_k|\phi_k\rangle\langle\phi_k|$ be an optimal decomposition for its entanglement of formation when $0 < \gamma < 1$. By Hughston, Jozsa and Wootters' theorem [20], there exists a $K \times K$ unitary matrix U, K being greater than or equal to $2n-3$, the rank of ϱ_γ, such that for each $k = 0, 1, \ldots, K-1$

$$\sqrt{p_k}|\phi_k\rangle = \sum_{i=0}^{1}\sum_{j=2}^{n-1} U^*_{k,(ij)}\sqrt{\frac{1-\gamma}{2(n-2)}}|ij\rangle + U^*_{k,2n-4}\sqrt{\gamma}|\psi^-\rangle \tag{16}$$

where $(ij) = i(n-2) + (j-2)$. Thus it can be obtained from equation (16) that

$$\begin{aligned} p_k\,\mathrm{tr}_B(|\phi_k\rangle\langle\phi_k|) &= \sum_{i,i'=0}^{1}\sum_{j=2}^{n-1} U^*_{k,(ij)}U_{k,(i'j)}\frac{1-\gamma}{2(n-2)}|i\rangle\langle i'| + \frac{1}{2}|U_{k,2n-4}|^2\gamma(|0\rangle\langle 0| + |1\rangle\langle 1|) \\ &= \frac{1-\gamma}{2(n-2)}\sum_{j=2}^{n-1}|\Psi_{kj}\rangle\langle\Psi_{kj}| + \frac{1}{2}|U_{k,2n-4}|^2\gamma(|0\rangle\langle 0| + |1\rangle\langle 1|) \end{aligned} \tag{17}$$

where

$$|\Psi_{kj}\rangle = \sum_{i=0}^{1} U^*_{k,(ij)}|i\rangle.$$

Then it follows from equation (17) and the concavity of $\mathcal{S}$ that

$$\begin{aligned} p_k E(|\phi_k\rangle) &= p_k\mathcal{S}(\mathrm{tr}_B\,|\phi_k\rangle\langle\phi_k|) \\ &\geqslant |U_{k,2n-4}|^2\gamma\mathcal{S}\left(\tfrac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)\right) \\ &= |U_{k,2n-4}|^2\gamma \end{aligned} \tag{18}$$

since the von Neumann entropy of a pure state has vanished. Hence, from equation (18) and the unitarity of U, we obtain the following inequality:

$$\begin{aligned} E_f(\varrho_\gamma) &= \sum_{k=0}^{K-1} p_k E(|\phi_k\rangle) \\ &\geqslant \sum_{k=0}^{K-1}|U_{k,2n-4}|^2\gamma \\ &= \gamma. \end{aligned} \tag{19}$$

By the two inequalities (15) and (19) we conclude that $E_f(\varrho_\gamma) = \gamma$.

We now consider the states corresponding to (α, γ) in the interior of the NPT region, that is, (α, γ) satisfying $1/2 < (n-2)\alpha + \gamma < 1$. By virtue of the convexity of the entanglement of formation, for a given (α, γ) the following inequality can be obtained:

$$\begin{aligned} E_f(\rho_{(\alpha,\gamma)}) &= E_f\left(\rho_{(t[1-N_{(\alpha,\gamma)}]/(n-2),\,t[2N_{(\alpha,\gamma)}-1]+(1-t)N_{(\alpha,\gamma)})}\right) \\ &\leqslant t\cdot E_f\left(\rho_{([1-N_{(\alpha,\gamma)}]/(n-2),\,2N_{(\alpha,\gamma)}-1)}\right) + (1-t)\cdot E_f\left(\rho_{(0,N_{(\alpha,\gamma)})}\right) \\ &= t\cdot[2N_{(\alpha,\gamma)}-1] + (1-t)\cdot\mathcal{E}(2N_{(\alpha,\gamma)}) \end{aligned} \tag{20}$$

where $N_{(\alpha,\gamma)} = (n-2)\alpha + \gamma$, and t is chosen by

$$\begin{aligned} \alpha &= t\cdot[1-N_{(\alpha,\gamma)}]/(n-2) \\ \gamma &= t\cdot[2N_{(\alpha,\gamma)}-1] + (1-t)\cdot N_{(\alpha,\gamma)}. \end{aligned} \tag{21}$$

Choosing the appropriate t satisfying equation (21), we are straightforwardly able to calculate the following upper bound on $E_f(\rho_{(\alpha,\gamma)})$:

$$\mathcal{E}(\mathcal{N}(\rho_{(\alpha,\gamma)})) + (n-2)\alpha\frac{2(n-2)\alpha + 2\gamma - 1 - \mathcal{E}(\mathcal{N}(\rho_{(\alpha,\gamma)}))}{1-(n-2)\alpha-\gamma}. \tag{22}$$

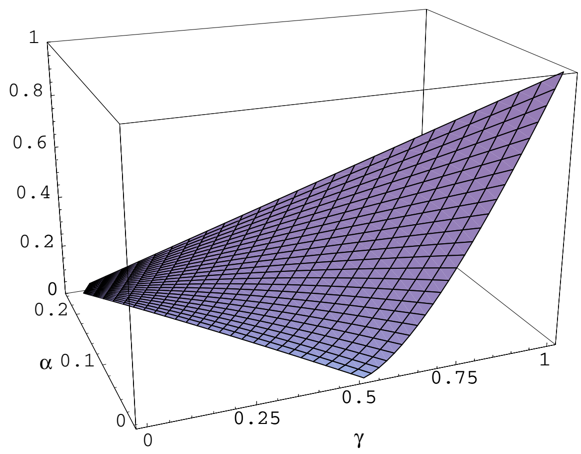

Figure 4. The upper bound (22) on the entanglement of formation of $\rho_{(\alpha,\gamma)}$ for $1/2 \leqslant (n-2)\alpha + \gamma \leqslant 1$ in a $2 \otimes 4$ quantum system.

For a $2 \otimes 4$ quantum system the upper bound (22) on $E_f(\rho_{(\alpha,\gamma)})$ is shown in figure 4. We remark that the entanglement of formation and its upper bound (22) have the same values for the states corresponding to three edges of the NPT region in figure 2. Therefore, we cannot derive any upper bound tighter than the upper bound (22) from the method using the convexity of the entanglement of formation as in equation (20). From this viewpoint, we can say that the upper bound (22) is a tight upper bound on $E_f(\rho_{(\alpha,\gamma)})$.

We note that $E_f(\varrho_\gamma) = \gamma = \mathcal{N}(\varrho_\gamma)$ and that $E_f(\rho_{(0,\gamma)}) \leqslant \mathcal{N}(\rho_{(0,\gamma)})$ for all $0 \leqslant \gamma \leqslant 1$. Thus, it follows that the entanglement of formation of any $\rho_{(\alpha,\gamma)}$ cannot exceed its negativity, that is,

$$E_f(\rho_{(\alpha,\gamma)}) \leqslant \mathcal{N}(\rho_{(\alpha,\gamma)})$$

for any $\rho_{(\alpha,\gamma)}$.

4. Conclusions

In this paper we exhibited a two-parameter class of states $\rho_{(\alpha,\gamma)}$ in a $2 \otimes n$ quantum system, found that an arbitrary state can be transformed into $\rho_{(\alpha,\gamma)}$ by means of LOCC and showed that $\rho_{(\alpha,\gamma)}$ is invariant under unitary operations of the form $U \otimes U$ on a $2 \otimes n$ quantum system. We finally investigated the entanglement for $\rho_{(\alpha,\gamma)}$ by computing two measures of entanglement, the negativity and the entanglement of formation.

For a higher dimensional quantum system, that is, an $m \otimes n$ quantum system $(m < n)$, we can exhibit a two-parameter class given by

$$\alpha \sum_{i=0}^{m-1} \sum_{j=m}^{n-1} |ij\rangle\langle ij| + \beta \left(\sum_{i,j=0(i<j)}^{m-1} \left|\varphi_{ij}^{+}\right\rangle\!\left\langle\varphi_{ij}^{+}\right| + \sum_{k=0}^{m-1} |kk\rangle\langle kk| \right) + \gamma \sum_{i,j=0(i<j)}^{m-1} |\varphi_{ij}^{-}\rangle\langle\varphi_{ij}^{-}| \qquad (23)$$

where

$$\left|\varphi_{ij}^{\pm}\right\rangle = \tfrac{1}{\sqrt{2}}(|ij\rangle \pm |ji\rangle)$$

and

$$m(n-m)\alpha + \frac{m(m+1)}{2}\beta + \frac{m(m-1)}{2}\gamma = 1.$$

Furthermore, we can show that any state in a $m \otimes n$ quantum system can be transformed into a state of the form of equation (23) by LOCC, and that any state in this class is $(U \otimes U)$-invariant for all unitary U in the group $G(m, n)$ defined in equation (2). Since every state in this class has properties analogous to those of $\rho_{(\alpha,\gamma)}$, one could investigate its entanglement.

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